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## A new case of integrability of the general Riccati equation and its application to relaxation problems

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**Abstract.** A new case of integrability by quadratures of the general Riccati equation is discussed. Some possibilities of its use in molecular physics are demonstrated by the example of vibrational-translational or rotational-translational relaxation of a small amount of diatomic molecules in a structureless gas.

The general Riccati equation (GRE), being tightly coupled to linear homogeneous second-order differential equations with variable coefficients (Kamke 1959), is widely used in different problems in theoretical physics and applied mathematics. That is why each case of its integrability is of particular interest. Let us consider therefore the following theorem.

*Theorem.* If in the GRE

$$y' = f(x)y^2 + g(x)y + h(x) \quad (1)$$

the coefficients  $f$ ,  $g$  and  $h$ , defined and continuous in some interval  $(a, b) \subset \mathbb{R}$ , are related as

$$f + g + h = \frac{d}{dx} \log \frac{\alpha}{\beta} - \frac{\alpha - \beta}{\alpha\beta} (\alpha f - \beta h) \quad (2)$$

with  $\alpha(x)$  and  $\beta(x)$  properly chosen functions differentiable in  $(a, b)$  such that  $\alpha\beta > 0$ , then this equation is integrable by quadratures.

*Proof.* The GRE is invariant with respect to a linear fractional transform of the type

$$y = \frac{\alpha z + \gamma}{\beta z + \delta} \quad (3)$$

with  $\Delta = \alpha\delta - \beta\gamma \neq 0$ , replacing (1) by an analogous equation for a new unknown function  $z(x)$  with coefficients

$$F = [\alpha^2 f + \alpha\beta g + \beta^2 h - (\alpha'\beta - \alpha\beta')]/\Delta \quad (4a)$$

$$G = [2\alpha\gamma f + (\alpha\delta + \beta\gamma)g + 2\beta\delta h - (\alpha'\delta - \alpha\delta') - (\gamma'\beta - \gamma\beta')]/\Delta \quad (4b)$$

$$H = [\gamma^2 f + \gamma\delta g + \delta^2 h - (\gamma'\delta - \gamma\delta')]/\Delta. \quad (4c)$$

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It is obvious that under the condition

$$F(x) \equiv 0 \quad (5)$$

this equation for  $z(x)$  is integrable by quadratures. The last condition requires that the coefficients of (1) should meet the relation

$$\alpha^2 f + \alpha \beta g + \beta^2 h = \alpha \beta \frac{d}{dx} \log \frac{\alpha}{\beta} \quad (6)$$

which is easily transformed to the form of (2) and that completes the proof of the theorem.

*Consequence 1.* If  $\alpha(x) \equiv \beta(x)$  or  $\chi \alpha(x) \equiv \zeta \beta(x)$  ( $\chi$  and  $\zeta$  constants) equation (6) reduces to the well known (Kamke 1959) cases of the integrability of GRE by quadratures:

$$f(x) + g(x) + h(x) \equiv 0 \quad (7)$$

$$\zeta^2 f(x) + \chi \zeta g(x) + \chi^2 h(x) \equiv 0. \quad (8)$$

*Consequence 2.* Equations (7) and (8) are obtained from (2) under the condition that the first term in the RHS equals zero. The other way to simplify (2) is to remove the second term taking  $\alpha/\beta = h/f$ . Then if  $f$  and  $h$  are differentiable in  $(a, b)$  and  $fh \neq 0$ , equation (2) yields

$$g = -(f+h) - (f'/f - h'/h). \quad (9)$$

This condition of integrability of the GRE is absent in Kamke's handbook (Kamke 1959). The general solution to (1) in this case is (provided  $\alpha = h$ ,  $\beta = f$ )

$$y = \frac{h(C + \int HE \, dx) + \gamma E}{f(C + \int HE \, dx) + \delta E} \quad E = \exp(-\int G \, dx) \quad (10)$$

with  $C$  being the integration constant. Functions  $G(x)$  and  $H(x)$  are defined by (4b) and (4c) respectively; functions  $\gamma(x)$  and  $\delta(x)$  are arbitrary.

*Consequence 3.* Equation (9) immediately gives that the GRE

$$y' = fy^2 + \hat{g}y + h - ys'/s \quad (11)$$

with  $f + \hat{g} + h \equiv 0$  is integrable by quadratures provided

$$sh/f = \text{constant}. \quad (12)$$

Now the general solution (10) transforms to

$$y = \frac{h[C + \int (f-g+h)D \, dx] - (f+h)D}{f[C + \int (f-g+h)D \, dx] + (f+h)D} \quad (13)$$

with  $g = \hat{g} + [\log(h/f)]'$ ,  $D = \exp[\int g(f-h)/(f+h) \, dx]$ , where the following coefficients in the transform (3) are taken:  $\alpha = h$ ,  $\beta = f$ ,  $\gamma = -1$ ,  $\delta = 1$ .

*Note.* Besides the case (5) the equation for  $z(x)$  is integrable also when  $H(x) \equiv 0$ . But comparing definitions (4a) and (4c) for the coefficients  $F$  and  $H$  it is an easy matter to show that this condition again gives relation (2) with  $\alpha$  replaced by  $\gamma$  and  $\beta$  by  $\delta$ . The same relation follows from the conditions of the type (7) and (8) on the coefficients  $F, G, H$  with substitutions  $\alpha \rightarrow \alpha + \gamma$ ,  $\beta \rightarrow \beta + \delta$  and  $\alpha \rightarrow \zeta\alpha + \chi\gamma$ ,  $\beta \rightarrow \zeta\beta + \chi\delta$  respectively.

*Example.* As an example let us consider the simplest case of the linear relaxation problem—the process of vibrational-translational or rotational-translational relaxation of a small amount of diatomic molecules in a structureless gas. In this case from a countable set of linear differential equations for the concentrations  $x_n$  of molecules in the  $n$ th internal state one can obtain the system for ‘adiabatic’ variables  $y_n$  introduced by the nonlinear transform (Dubrovskiy, Strelchenya 1986)

$$x_n/x_{n-1} = s_n y_n, \quad s_n = K_{n-1,n}/K_{n,n-1} \quad n \geq 1 \quad (14)$$

with  $K_{m,n}$  being the inelastic  $m \rightarrow n$  transition rate constants. This system (in the single quantum approximation) has the form

$$y'_n = f_n y_n^2 + g_n y_n + h_n + y_n [e_n(y) - s'_n/s_n] \quad (15)$$

with

$$\begin{aligned} f_n &= K_{n,n+1} - K_{n-1,n} & h_n &= K_{n,n-1} - K_{n-1,n-2} \\ g_n &= -(f_n + h_n) \\ e_n &= y_n K_{n,n+1} \left( \frac{y_{n+1}}{y_n} - 1 \right) - \frac{1}{y_n} K_{n-1,n-2} \left( \frac{y_n}{y_{n-1}} - 1 \right). \end{aligned} \quad (16)$$

For distributions gradual in  $n$  ( $y_n \sim y_{n+1}$ ) the non-diagonal terms  $e_n(y)$  are small and (provided they are ignored) system (15) splits into the set of uncoupled equations of the type (11) with  $f_n + g_n + h_n \equiv 0$  for every  $n$ . (That is the reason why the transform (14) may be considered as an analogue of the transition from phase to adiabatic variables in Hamiltonian mechanics (ter Haar 1971) while the variables  $y_n$  themselves may be called adiabatic ones.)

In the case of relaxation of harmonic oscillators, for example, with  $K_{n,n+1} = (n+1)K_{0,1}$ ,  $K_{n+1,n} = (n+1)K_{1,0}$  the relation (12) is obviously met and the time evolution of the concentrations can be described analytically (by quadratures)

$$x_n(t) = x_0 \prod_{i=1}^n s_i y_i(t) \quad n \geq 1 \quad (17)$$

with  $y_i(t)$  defined by the general solution (13). The same situation holds for the relaxation of slightly anharmonic oscillators with  $K_{n,n+1} = (n+1)K_{0,1}\theta^n$ ,  $K_{n+1,n} = (n+1)K_{1,0}\theta^n$  provided  $\theta = 1 + \varepsilon$ , where  $\varepsilon$  is small adiabatic correction. In this case the relation (12) is met up to terms of second order in  $\varepsilon$ . Finally, such an approach leads to an exactly solvable problem of rotational relaxation of molecules when the exponential gap rates are used ( $K_{n+1,n} = A \exp[-B\Delta E_{n+1,n}]$ ).

A more detailed treatment of all of these problems is the subject of a separate paper which will be published elsewhere.

## References

- Dubrovskiy G V and Strelchenya V M 1986 *Zh. Prikl. Mech. Techn. Fiz.* 3 22-31 (in Russian)  
 Kamke E 1959 *Differential Gleichungen: Lösungsmethoden und Lösungen* (Leipzig: Teubner)  
 Ter Haar D 1971 *Elements of Hamiltonian Mechanics* (Oxford: Pergamon)